We have yet to express $-\frac{\partial^2 \Omega}{\partial N^2}$ in terms of the differentials of u and s.

Now $-\frac{1}{2}D\varpi = D\Omega = \Omega_u Du + \Omega_s Ds$.

Hence
$$
\begin{aligned}
-\tfrac{1}{2}D^2\varpi &= \Omega_u D^2u + \Omega_s D^2 s \\
&\quad + \Omega_{uu}(Du)^2 + 2\Omega_{us}DuDs + \Omega_{ss}(Ds)^2 \\
&= -\Omega_u(2mDu + 2\Omega_s) + \Omega_s(2mDs - 2\Omega_u) \\
&\quad + \varpi\{e^{2i\psi}\Omega_{uu} + 2\Omega_{us} + e^{-2i\psi}\Omega_{ss}\} \\
&= -2m\varpi(2m + iD\psi) - 4\Omega_u\Omega_s \\
&\quad + \varpi\{e^{2i\psi}\Omega_{uu} + 2\Omega_{us} + e^{-2i\psi}\Omega_{ss}\}
\end{aligned}
$$

Also
$$
\begin{aligned}
4\Omega_u\Omega_s &= \{(\Omega_u Du + \Omega_s Ds)^2 - (\Omega_u Du - \Omega_s Ds)^2\}/\varpi \\
&= \frac{1}{\varpi}(D\Omega)^2 - \varpi(2m + iD\psi)^2
\end{aligned}
$$

and this gives on substitution

$$
\begin{aligned}
e^{2i\psi}\Omega_{uu} &- 2\Omega_{us} + e^{-2i\psi}\Omega_{ss} \\
&= -\frac{1}{2\varpi}D^2\varpi + \frac{1}{4\varpi^2}(D\varpi)^2 + m^2 - (m + iD\psi)^2 - 4\Omega_{us} \\
&= -D\left(\frac{D\varpi}{2\varpi}\right) - \left(\frac{D\varpi}{2\varpi}\right)^2 + m^2 - (m + iD\psi)^2 - 4\Omega_{us}
\end{aligned}
$$

and since
$$\Omega = \frac{K}{(us)^{\frac{1}{2}}} + \tfrac{3}{8}m^2(u+s)^2,$$
$$\Omega_{us} = \tfrac{3}{4}\frac{K}{r^3} + \tfrac{3}{4}m^2,$$

whence

$$\Theta = -\left(\frac{K}{r^3} + m^2\right) + 2(m + iD\psi)^2 - D\left(\frac{D\varpi}{2\varpi}\right) - \left(\frac{D\varpi}{2\varpi}\right)^2,$$

which is Hill's form.

On the Construction of Telescopes whose Relative or Absolute Focal Length shall be Invariable at all Temperatures. By F. L. O. Wadsworth.

(*Communicated by the Secretaries*)

In the case of all optical instruments which are designed for the accurate measurement of either relative or absolute positions it is extremely necessary to avoid the necessity for refocussing the instrument for changes of temperature. Such refocussing is likely to produce both errors of measurement and errors of adjustment. The first have already been dealt with in

a recent paper,* where the effect of refocussing is considered especially with reference to spectrometric, heliometric, and micrometric observations.† The second class of errors are more purely mechanical in their nature, and apply more particularly to measurements with the transit, meridian circle, zenith telescope, &c. Here the effect is to disturb the position of the axis of collimation, and may, of course, be eliminated by a redetermination of that constant. Such disturbances and readjustments are, however, to be avoided whenever possible. My attention was first called to this problem in working out the design of the proposed 6″ photographic transit and meridian circle for the new Allegheny Observatory,‡ and more recently by the experiences of Professor Updegraff with the new 6″ meridian circle of the U.S. Naval Observatory. Professor Updegraff found that the apparent focal length of this instrument was decidedly different at different seasons of the year. No such result had ever before been observed with any of the other similar instruments at the observatory, and Professor Updegraff stated to me that some of the observers there disputed and even ridiculed the idea of such a change taking place; but, as a matter of fact, both the variability of the apparent focal length of the 6″ and the invariability of the focal length of the other instruments are fully explained, and might have been predicted, by theory. There are a few points of interest in this connection which it has seemed desirable to develop.

The variability of the focal length of a refracting objective at different temperatures has been long known, but the cause of this variability does not seem to be well understood. The theory of the subject seems to have been first investigated by Krueger.§ Later, Sundell made an extended series of measurements of the focal length of the Helsingfors heliometer, and found ‖ that the results obtained differed by more than 20 per cent. from those given by Krueger's formulæ. It was immediately pointed out by Professor Hastings ¶ that this difference was probably due in large part to the error in Professor Krueger's assumption that the "refractive power" of a glass varies directly as the density. This assumption is now well known to be incorrect. The expression derived by Professor Hastings for the variation, ΔF, of the focal length of an objective for a difference in temperature, ΔT, is as follows:

$$\Delta F = -F^2\left[\frac{dn_1}{dt}A + (n_1 - 1)\frac{dA}{dt} + \frac{dn_2}{dt}B + (n_2 - 1)\frac{dB}{dt}\right]\Delta T \quad (1)$$

* "On the Optical Conditions required to secure Maximum Accuracy of Measurement in the Use of the Telescope and Spectroscope," *Astrophysical Journal*, vol. xvi. pp. 267–299; vol. xvii. pp. 1–19, 100–132.

† See particularly vol. xvii. pp. 7–19.

‡ See Report of the Director, 1900, pp. 8 and 29; *ibid.* 1901, p. 16.

§ *Astronomische Nachrichten*, Bd. 60, s. 65.

‖ *Ibid.* Bd. 103, s. 19–26; also Bd. 111, s. 257–262.

¶ *Ibid.* Bd. 105, s. 69–72.

where n_1 and n_2 are the indices of refraction of the crown and flint glasses for the wave-length corresponding to any given spectral focus of the objective, and A, B are the sums of the curvatures of the surfaces of the two lenses. All four differential coefficients, $\frac{dn}{dt}$ and $\frac{dA}{dt}$, must be determined by experiment.

The above expression (1) is derived directly from the differentiation of the usual expression for the focal length of a compound (achromatic) lens, and neglects the effect of thickness and separation of the two lenses. Ordinarily the effect of these quantities on the differential changes of focal length with temperature will be themselves differentials of a second order, and may be neglected. Within these limits therefore (1) is rigorously correct, provided the refraction coefficients $\frac{dn}{dt}$ are properly interpreted and measured. This last precaution, however, is an important one, and does not seem to have always been regarded. Thus Professor Hastings himself, in applying the formula to the calculation of the focal length of a given lens, seems to have used for $\frac{dn_1}{dt}$ and $\frac{dn_2}{dt}$ values derived from the measurements of variations in the *relative* indices of glass and air. This is correct only when the temperature of the lens is the *same* as that of the air and both change uniformly. Generally this last condition is not fulfilled, and we must then express $\frac{dn}{dt}$ with reference not to the *relative* index of refraction n, but with reference to the *absolute* indices, n_0 and N_0, of glass and air respectively.

From the well-known relation

$$n = \frac{n_0}{N} \qquad \dots \qquad \dots \qquad \dots \qquad (2)$$

we have

$$\frac{dn}{dt} = \frac{1}{N}\frac{dn_0}{dt} - \frac{n_0}{N^2}\frac{dN}{dt} \simeq \frac{dn_0}{dt} - n\frac{dN}{dt} \qquad \dots \qquad (3)$$

since N is itself very nearly equal to unity at all temperatures.

The value of the second term of (3) for any particular glass is nearly constant. According to the most recent determinations, the thermal refractive coefficient for air at ordinary pressure and temperature (760 mm. and 20° C.) is

$$\frac{dN}{dt} \simeq -0{\cdot}93 \times 10^{-6} \qquad \dots \qquad \dots \qquad \dots \qquad (4)$$

For the flint and crown glasses, Feil 1237 and Feil 1219,

investigated by Professor Hastings, the relative temperature coefficients for wave-lengths 5600 are*

$$\left.\begin{array}{l}\text{Crown } 1219, \dfrac{dn_1}{dt} = +0{\cdot}1 \times 10^{-6} \\ \text{Flint } 1237, \dfrac{dn_2}{dt} = +5{\cdot}4 \times 10^{-6}\end{array}\right\} \quad \ldots \quad \ldots \quad (5)$$

The relative indices, n, for these two glasses for the same wave-lengths are

$$n_1 = n_{1219} = 1{\cdot}519+$$
$$n_2 = n_{1237} = 1{\cdot}628$$

Hence the absolute temperature coefficients $\frac{dn_0}{dt}$ are found to be

$$\text{For crown } 1219, \frac{dn_0}{dt} \cong -1{\cdot}31 \times 10^{-6}$$

$$\text{For flint } 1237, \frac{dn_0}{dt} \cong +3{\cdot}9 \times 10^{-6}$$

The effect on the focal length of a given change in the temperature of the air in the telescope tube is therefore very nearly equal to the effect of the same change in the temperature of the crown glass, and is not quite one-third as great as that produced by an equal temperature variation in the flint. In the first case the effects are nearly compensatory; in the second case they are additive. Hence any difference in the temperature variation, Δt, of the air and the objective will produce a decided difference in the quantity $\frac{dn}{dt}$ which appears in (1) and (3), and will lead to a corresponding difference in ΔF. In general the temperature change in the glass will lag behind that in the surrounding air; hence, if the temperature is rising the quantity $\frac{dn}{dt}$ will be smaller than would be indicated by measurement, and the observed focal length will be shorter than that calculated for the given air temperature; if the temperature is falling, the reverse will be the case, and the measured focal length will be longer than the calculated. This fact may explain the discrepancies that have been sometimes observed in measuring the variations of focal length with temperature.

The magnitude of the differential change in the focal length of a given telescope is dependent upon the nature of the glass employed in its construction. For the purpose of comparison I have collected the results of measurements on a number of different object-glasses of different sizes and types of construction, which are presented in the following table :—

* *American Journal of Science*, vol. xv. p. 269.

TABLE I.

Instrument.			Change in F.				Conditions of Observation.		
Objective.	A	F.	ΔT(C).	ΔF (apparent).	ΔZ (tube).	Total $\frac{\Delta F}{F \Delta T} = a''$.	Observer.	Reference.	Remarks.
	cm.	cm.	°	cm. cm.	cm.				
Helsingfors	...	...	28·1	0·177		2·19 × 10−5	Sundell	Ast. Nach. Vol. ciii. p. 24	Winter of 1879 16°·2 C. to −11°·9 C.
Heliometer	...	298·3	31·9	0·211		2·29 × 10−5	Sundell	Vol. cxi. p. 261	Winter of 1881 17°·3 C. to −14°·6 C.
Strassburg Heliometer	...	691·6	27·8	1·08		5·1 × 10−5	Schur	Ast. Nach. Vol. cxix. p. 249	+24°·6 C. to −3°·2 C.
Göttingen Heliometer	...	...	About 40	...		2·06 × 10−5	Schur	Ambronn. Handbuch	—
Emerson McMillen	30·5	457·2	27·5	0·18	0·13*	2·46 × 10−5	Lord	Astrophysical Journal, Vol. v. p. 305	Change measured for H_β
Halstead	58·4	917·7	30·0	Very small	0·22†	1·17 × 10−5	Young	From unpublished observations	Never very carefully observed
Washington	66·0	990·0	About 24·4	0·24	0·27†	2·11 × 10−5	See and Dinwiddie	Report U.S. Naval Obs. 1902	Visual change
Lick	91·4	1763· +	30	1·7	0·6*	4·3 × 10−5	Campbell	Astrophysical Journal, Vol. viii. p. 138	Change measured for H_γ
Yerkes	102·0	1936· +	50	0·72	0·97*	1·64 × 10−5	Hale and Ellermann	Astrophysical Journal, Vol. x. p. 93.	Not stated, probably visual

* Computed for steel tube. † Computed for combination tube (steel and brass).

The wide variations in the absolute values of the temperature co-efficients α'' is remarkable, and seems to indicate that little attention has been paid to the question of controlling or eliminating changes in focal length. But as already stated, this question is an important one in the case of instruments designed for accurate measurements, and should in such cases be quite as carefully considered as any of the other optical problems connected with the design of objectives.

In general, the condition which we desire to fulfil is that the apparent focal length of the instrument shall remain unaltered. This means that the coefficient of change in focal length α'' shall be equal to the coefficient of linear expansion α_t of the metal of which the tube of the telescope is constructed, *i.e.*

$$\alpha'' = \frac{\Delta F}{F} = \alpha_t = \frac{\Delta Z}{Z} \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

Equation (1) may be put in the form

$$\frac{\Delta F}{F} = -F\left\{\frac{dn_1}{dt}A + \frac{dn_2}{dt}B - \alpha_g\left[(n_1 - 1)A + (n_2 - 1)B\right]\right\}\Delta t$$

or for one degree change in temperature, $(\Delta t = 1)$

$$\frac{\Delta F}{F} = \alpha'' = \alpha_g - F\left(\frac{dn_1}{dt}A + \frac{dn_2}{dt}B\right) \quad \ldots \quad (7)$$

where α_g is the coefficient of linear expansion of the glass, which is assumed to be the same for both flint and crown.

In order to satisfy the conditions of achromatism, we must also have for A and B—

$$\left.\begin{aligned} A &= \frac{1}{F\Delta n_1\left[\dfrac{n_1 - 1}{\Delta n_1} - \dfrac{n_2 - 1}{\Delta n_2}\right]} \\ B &= \frac{1}{F\Delta n_2\left[\dfrac{n_2 - 1}{\Delta n_2} - \dfrac{n_1 - 1}{\Delta n_1}\right]} \end{aligned}\right\} \quad \ldots \quad \ldots \quad (8)$$

Substituting these values of A and B in (7) we obtain

$$\left.\begin{aligned} \alpha'' &= \alpha_g - \frac{\dfrac{dn_1}{dt}}{\Delta n_1\left[\dfrac{n_1 - 1}{\Delta n_1} - \dfrac{n_2 - 1}{\Delta n_2}\right]} - \frac{\dfrac{dn_2}{dt}}{\Delta n_2\left[\dfrac{n_2 - 1}{\Delta n_2} - \dfrac{n_1 - 1}{\Delta n_1}\right]} \\ &= \alpha_g + \frac{1}{\dfrac{n_1 - 1}{\Delta n_1} - \dfrac{n_2 - 1}{\Delta n_2}}\left[\frac{1}{\Delta n_2}\frac{dn_2}{dt} - \frac{1}{\Delta n_1}\frac{dn_1}{dt}\right] \end{aligned}\right\} \quad (9)$$

and the problem is to choose two glasses (crown and flint) whose

optical properties are such as to make the numerical value of α'' from (9) satisfy the condition expressed in (6).

Through the researches of Fizeau,* Hastings,† Müller,‡ Vogel,§ Pulfrich,|| Reed,¶ and Winkelmann** we now have available the constants α_g, n, Δn, and $\frac{dn}{dt}$ for a considerable number of flint and crown glasses. These values are collected together in the following table :—

* *Annales de Chimie et de Physique* (3), vol. lxvi. p. 429.
† *American Journal of Science* (3), vol. xv. p. 269.
‡ *Publicationen des Astrophysicalischen Observatoriums zu Potsdam*, vol. iv. p. 151.
§ Wiedmann, *Annalen*, vol. xxv. p. 87.
|| *Ibid.* vol. xlv. p. 609.
¶ *Inaugural-Dissertation*, Jena, 1897.
** Wiedmann, *Annalen*, 1892 to 1897.